Equilibrium sets in quintom cosmologies: the past asymptotic dynamics

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In the previous paper [7] was investigated the phase space of quintom cosmologies for a class of exponential potentials. This study suggests that the past asymptotic dynamics of such a model can be approximated by the dynamics near a hyperbola of critical points. In this paper we obtain a normal form expansion near a fixed point located on this equilibrium set. We get computationally treatable system up to fourth order. From the structure of the unstable manifold (up to fourth order) we see that that the past asymptotic behavior of this model is given by a massless scalar field cosmology for an open set of orbits (matching the numerical results given in [7]). We complement the results discussed there by including the analysis at infinity. Although there exists unbounded orbits towards the past, by examining the orbits at infinity, we get that the sources satisfy the evolution rates $\dot\phi^2/V \sim \frac{2m^2}{n^2-m^2}$, $\dot\phi/\dot\varphi \sim -m/n$, with $H/\dot\phi$ approaching zero.

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I. INTRODUCTION

Since 1998, the observational evidence coming from type I-a Supernovae (SNIa), Large Scale Structure (LSS) formation, and the Cosmic Microwave Background (CMBR) Radiation [1] for an expanding Universe is more and more convincing. The acceleration of the expansion seems to be fuelled by a yet unknown kind of matter named dark energy (see the reviews [2]). The most promising particle candidates for dark energy are spin zero bosons, which in the mathematical formalism of Quantum Field Theory are represented by scalar fields. Many models of dark energy use a single scalar field and two very popular options are the so called phantom field and the quintessence field.

Beyond the theories with a single scalar field, models with two fields (quintessence and phantom) have settled out explicitly and named quintom models [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The quintom paradigm is a hybrid construction of a quintessence component, usually modelled by a real scalar field that is minimally coupled to gravity, and a phantom field: a real scalar field –minimally coupled to gravity– with negative kinetic energy. Let us define the equation of state parameter of any cosmological fluid as $w \equiv$ pressure/density. The simplest model of dark energy (vacuum energy or cosmological constant) is assumed to have $w = -1$. A key feature of quintom-like behavior is the crossing of the so called phantom divide, in which the equation of state parameter crosses through the value $w = -1$. Quintom behavior (i.e., the $w = -1$ crossing) has been investigated in the context of h-essence cosmologies [4, 5]; in the context of holographic dark energy [14]; inspired by string theory [9]; derived from spinor matter [10]; for arbitrary potentials [12, 13]; using isomorphic models consisting of three coupled oscillators, one of which carries negative kinetic energy (particularly for investigating the dynamical behavior of massless quintom)[19]. The crossing of the phantom divide is also possible in the context of scalar tensor theories [20, 21, 22, 23, 24] as well as in modified theories of gravity [25].

The cosmological evolution of quintom model with exponential potential has been examined, from the dynamical systems viewpoint, in [6] and [7, 8]. The difference between [6] and [7, 8] is that in the second case the potential considers the interaction between the conventional scalar field and the phantom field. In [7] the case in which the interaction term dominates against the mixed terms of the potential, was studied. It was proven there, as a difference with the results in [8], that the existence of scaling attractors (in which the energy density of the quintom field and the energy density of DM are proportional) excludes the existence of phantom attractors. Some of this results were extended in [12], for arbitrary potentials.

In [7] were investigated curves of (non-hyperbolic) critical points $C_\pm$, and other types of hyperbolic critical points from the dynamical systems perspective. We will focus here on the curves $C_\pm$. These curves are associated with the past dynamics of quintom models [7]. This fact was supported by numerical integrations. Although can be argue that these curves are non-physical because the value $w = 1$, we believe that the investigation of the past dynamics of such system is important for completeness. This involve the analysis of sets of non-hyperbolic critical points.

When a vector field $\mathbf{x}' = \mathbf{X}(\mathbf{x})$ in $\mathbb{R}^n$, admits non-isolated critical points, for instance a curve C of critical points, then the matrix of derivatives $\mathbf{DX}(\mathbf{x})$, when evaluated on each point in the equilibrium set C, has necessarily one zero eigenvalue. Hence, each point in C is non-hyperbolic. Then, linealization technique fails to be applied. Although the equilibrium points in C are

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non-hyperbolic, one is able to apply the Invariant Manifold Theorem [26, 27, 28] to guaranteed conditions for the existence of stable and unstable manifolds. Suppose that each point in the equilibrium set C, assumed to be a curve, has an unstable manifold of dimension n_u, the union of these manifolds forms an (n_u+1)-dimensional set whose orbits approach a point of C as $t \to -\infty$. If $n_u = n-1$ we say that C is a local source [26]. If the equilibrium set has only one zero eigenvalue and the others with non-zero real parts, then the equilibrium set is called normally hyperbolic [29]. In such a case one is able to obtain useful information about the stability of the critical set by examining the signs of the other, non-null, eigenvalues, as we will see later. However, if one is interested in the construction of center manifolds, or at least approximated ones, if conditions for their existence are fulfilled (for references on center manifolds see [27], chapter 4; [30], [28](b), chapter 18), we require to use other tools, e.g., normal forms.

For the construction of (at least, approximated) invariant manifolds for critical points, we can transform a dynamical system defined in the neighborhood of a critical point to a non-linear dynamical system in its simpler form. The theory of Normal Forms (NF) can be applied in order to do so, and in most occasions, terms of order higher than two in the Taylor expansion are required. For the construction of normal forms for vector fields in $\mathbb{R}^n$ we will follow the approach in [27], chapter 2. NF calculations consist of two stages: first, the construction of the normal forms in which case the nonlinear terms take their simplest form and second, the determination of the topological types of the fixed points from the normal forms. Then by using techniques described in [28](b), we will able to construct approximated invariant manifolds up to the desired order in the vector norm.

In this paper we try and investigate Normal Forms expansions of the system in [7] near an hyperbolae of critical points. Our main purpose is to devise approximated unstable and center manifolds. We develop the normal expansion (with undetermined coefficients) up to an arbitrary order. The system resulting by truncating error terms admits a solution given in quadratures, which in general cannot be solved explicitly. By integrating the resulting system, one is able to construct (theoretical) approximated unstable and center manifolds. The problem to determine all the coefficient values, up to a prescribed order, if very difficult. Our computing power allow us to determine the coefficients values up to fourth order.

The paper has been organized as follows: in section II we make a general description of the model under study. In section III we recall and improve some results from reference [7]. To make the paper self-contained we explicitly state in section A the main techniques of the NF expansions by following the approach in [27] (see also [28](b), chapter 19). We apply these techniques to the quintom scalar field solution in section IV (which we argue are associated to the past asymptotic dynamics). In section V we offer the analysis at infinity obtaining sufficient conditions for the existence of local sources at this regime. In section VI we offer our main conclusions.

II. THE QUINTOM MODEL

In the following discussion on the quintom phase space analysis we restrict ourselves to the two-field quintom model, with a Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\phi,\varphi), \tag{1}$$

and we include, also, ordinary matter (a comoving perfect fluid) in the gravitational action. As in [7] we consider here the efective two-field potential

$$V = V_0 e^{-\sqrt{6}(m\phi+n\varphi)}, \tag{2}$$

where the scalar field ϕ represents quintessence and φ represents a phantom field. For simplicity, we assume $m>0$ and $n>0$.

The geometry is given by the Friedmann-Robertson-Walker (FRW) metric (in spherical coordinates):

$$ds^2 = -dt^2 + a(t)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\vartheta^2\right). \tag{3}$$

The field equations derived from (3), are

$$H^2 - \tfrac{1}{6}\left(\dot\phi^2 - \dot\varphi^2\right) - \tfrac{1}{3}V - \tfrac{1}{3}\rho_{\rm M} = 0, \tag{4}$$

$$\dot H = -H^2 - \tfrac{1}{3}\left(\dot\phi^2 - \dot\varphi^2\right) + \tfrac{1}{3}V - \tfrac{1}{6}\rho_{\rm M}, \tag{5}$$

$$\dot\rho_{\rm M} = -3H\rho_{\rm M}, \tag{6}$$

$$\ddot\phi + 3H\dot\phi - \sqrt{6}mV = 0, \tag{7}$$

$$\ddot\varphi + 3H\dot\varphi + \sqrt{6}nV = 0, \tag{8}$$

where $H = \frac{\dot a(t)}{a(t)}$ denotes de Hubble expansion scalar.

The dot denotes derivative with respect the time t.

III. SOME RESULTS FOR THE FLAT FRW CASE

In [7] it was studied an homogenous and isotropic universe, having dust (DM) and quintom DE using the standards tools of the theory of dynamical systems. There are introduced the normalized variables: $(x_\phi,\, x_\varphi,\, y)$, defined by

$$x_\phi = \frac{\dot\phi}{\sqrt{6}H},\; x_\varphi = \frac{\dot\varphi}{\sqrt{6}H}, y = \frac{\sqrt{V}}{\sqrt{3}H}. \tag{9}$$

They are related through the Friedman equation (4) by $x_\phi^2 - x_\varphi^2 + y^2 = 1 - \frac{\rho_{\rm M}}{3H^2} \leq 1$.

The dynamics in a phase space is governed by the vector field (or differential equation) [7]:

$$x'_\phi = \frac{1}{3}\left(3my^2 + (q-2)x_\phi\right) \tag{10}$$

$$x'_\varphi = -\frac{1}{3}\left(3ny^2 - (q-2)x_\varphi\right) \tag{11}$$

$$y' = \frac{1}{3}(1 + q - 3(mx_\phi + nx_\varphi))y \tag{12}$$

defined in the phase space given by

$$\Psi = \{\mathbf{x} = (x_\phi, x_\varphi, y) : 0 \le x_\phi^2 - x_\varphi^2 + y^2 \le 1\}. \tag{13}$$

Here the prime denotes differentiation with respect to a new time variable $\tau = \log a^3$, where a is the scale factor of the space-time. The deceleration factor $q \equiv -\ddot{a}a/\dot{a}^2$ can be written as

$$q = \frac{1}{2}\left(3\left(x_\phi^2 - x_\varphi^2 - y^2\right) + 1\right). \tag{14}$$

A. Linear analysis of non-isolated critical points

In this section we want to investigate, in more detail, the behavior of the two curves (hyperbolae) of non-isolated fixed points $C_\pm$ with coordinates $x_\phi = \pm\sqrt{1 + {x_\varphi^*}^2}$, $x_\varphi = x_\varphi^*$, $y = 0$.

From the physical viewpoint, the curves $C_\pm$ represent solutions in which the contribution of matter and the potential energy to the total energy density is negligible. These solutions are therefore of stiff-fluid type ($w = 1$), which in turn correspond to a decelerating universe. On the other hand, the curves $C_\pm$ are local sources, and therefore it is unlikely that they can represent the final stage in the evolution of our Universe. Thus, by analyzing the sign of the real part of the normally-hyperbolic curves $C_\pm$ we get the following results (we are assuming $m > 0$ and $n > 0$):

1. If $m < n$, C_+ contains an infinite arc parameterized by x_φ^* such that $x_\varphi^* < \frac{-n-m\sqrt{1-m^2+n^2}}{m^2-n^2}$ that is a local source. C_- contains an infinite arc parameterized by x_φ^* such that $x_\varphi^* < \frac{-n+m\sqrt{1-m^2+n^2}}{m^2-n^2}$ that is a local source.

2. If $m = n$, C_+ contains an infinite arc parameterized by x_φ^* such that $x_\varphi^* < \frac{1-m^2}{2n}$ that is a local source. All of C_- is a local source.

3. If $m > n$ there are two possibilities

   (a) If $m^2 - n^2 < 1$, all of C_- is a local source. A finite arc of C_+ parameterized by x_φ^* such that $\frac{-n-m\sqrt{1-m^2+n^2}}{m^2-n^2} < x_\varphi^* < \frac{-n+m\sqrt{1-m^2+n^2}}{m^2-n^2}$ is a local source.

   (b) If $m^2 - n^2 \ge 1$, no part of C_+ is a local source and all of C_- is a local source.

In conclusion, we have that C_- is always a local source provided $m \ge n > 0$. But, we said nothing about its global stability. To answer this question, one needs to apply more sophisticated tools such as the calculations of Normal Forms (we submit the reader to section IV for such an investigation).

B. Heteroclinic sequences

From the local stability properties of $C_\pm, O, P$ and T and supported by several numerical integrations [7] it was possible to identify heteroclinic sequences (see table I for additional information concerning the critical points $O, C_\pm, T, P$ reported in [7]).

- Case i a) If $0 \le m < m$, the point P is an stable node and T does not exists. For this conditions there exist an heteroclinic sequence of type $\mathcal{K}_- \longrightarrow O \longrightarrow P$ or of type $\mathcal{K}_+ \longrightarrow O \longrightarrow P$, where $\mathcal{K}_\pm$ are infinite arcs contained in $C_\pm$.

- Case i b) If $0 < n = m$, the point P is an stable node and T does not exists. There exists an heteroclinic sequence of type $C_- \longrightarrow O \rightarrow P$ or of type $\mathcal{K}_+ \longrightarrow O \longrightarrow P$.

- Case i c) If $0 < n < m < \sqrt{n^2 + 1/2}$, the point P is an stable node and T does not exists. There exists an heteroclinic sequence of type $C_- \rightarrow O \rightarrow P$ or of type $\kappa_+ \longrightarrow O \longrightarrow P$, where κ_+ is a finite arc contained in C_+.

- Case ii) For $n > 0$ and $\sqrt{n^2 + 1/2} < m \le \sqrt{n^2 + 4/7}$, the point T is a stable node and the point P is a saddle. For these conditions the heteroclinic sequence is of type $C_\pm \longrightarrow O \longrightarrow T \longrightarrow P$, or of type $\kappa_+ \longrightarrow O \longrightarrow T \longrightarrow P$.

- Case iii) For $n > 0$ and $\sqrt{n^2 + 4/7} < m < \sqrt{1 + n^2}$, the point T is a spiral node and the point P is a saddle. For these conditions the heteroclinic sequence is the same as in the former case.

- Case iv) For $n > 0$ and $m > \sqrt{1 + n^2}$ the point T is a spiral node whereas the point P does not exist. The heteroclinic sequence in this case is $C_- \longrightarrow O \longrightarrow T$.

IV. NORMAL EXPANSION UP TO ARBITRARY ORDER

Normal Form (NF) calculations essentially removes the quadratic, cubic, etc., terms that are effectively neglectful up to the desired order in the Taylor expansion around a non-hyperbolic fixed point, e.g., $C_\pm$. The terms that cannot be removed by using NF calculations, and they are the essential degrees of nonlinearity (see appendix

TABLE I: Location, existence and deceleration factor of the critical points for $m>0$, $n>0$ and $y>0$. We use the notation $\delta = m^2 - n^2$ (from reference [7]).

Name	x_ϕ	x_φ	y	Existence	q
O	0	0	0	All m and n	$\frac{1}{2}$
$C_\pm$	$\pm\sqrt{1+x_\varphi^{*\,2}}$	x_φ^*	0	All m and n	2
P	m	$-n$	$\sqrt{1-\delta}$	$\delta<1$	$-1+3\delta$
T	$\frac{m}{2\delta}$	$-\frac{n}{2\delta}$	$\frac{1}{2\sqrt{\delta}}$	$\delta \geq 1/2$	$\frac{1}{2}$

IV). In this section we obtain the normal form expansion for nonhyperbolic points in the curve C_-.

The normal expansion up to order N is given by the

Proposition IV.1 *Let be the vector field* $\mathbf{X}$ *given by (10-12) which is* C^∞ *in a neighborhood of* $\mathbf{x}^* = (x_\phi^*, x_\varphi^*, y^*)^T \in C_-$. *Let* $m \geq n > 0$, *and* $x_\varphi^* \in \mathbb{R}$, *such that* $\lambda_3^- = 1 - n x_\phi^* + m\sqrt{1+x_\phi^{*\,2}}$ *is not integer, then, there exist constants* a_r, b_r, c_r, $r \geq 2$ *(non necessarily different from zero) and a transformation of coordinates* $\mathbf{x} \rightarrow \mathbf{y}$, *such that (10-12) has normal form*

$$y_1' = \sum_{r=2}^{N} a_r y_1^r + O(|\mathbf{y}|^{N+1}), \quad (15)$$

$$y_2' = y_2\left(1 + \sum_{r=2}^{N} b_r y_1^{r-1}\right) + O(|\mathbf{y}|^{N+1}), \quad (16)$$

$$y_3' = y_3\left(\lambda_3 + \sum_{r=2}^{N} c_r y_1^{r-1}\right) + O(|\mathbf{y}|^{N+1}), \quad (17)$$

defined in neighborhood of $\mathbf{y} = (0,0,0)$.

Proof. By applying linear coordinate transformations one is able to reduce the linear part of $\mathbf{X}$ to the form

$$\mathbf{X}_1(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{J}\mathbf{x}.$$

By the hypothesis $m \geq n > 0$ we guaranteed $\lambda_3^- > 1$ for all $x_\varphi^* \in \mathbb{R}$. Since, the eigenvalues of $\mathbf{J}$ are different and $\mathbf{J}$ is diagonal; then, the corresponding eigenvectors

$$B = \left\{ x_1^{m_1} x_2^{m_2} x_3^{m_3} \mathbf{e}_i | m_j \in \mathbb{N}, \sum m_j = r, i, j = 1,2,3 \right\}$$

form a basis of H^r. Since

$$\mathbf{L_J} \mathbf{x}^{\mathbf{m}} \mathbf{e}_i = \{(\mathbf{m}\cdot\lambda) - \lambda_i\} \mathbf{x}^{\mathbf{m}} \mathbf{e}_i,$$

the eigenvectors in B for which $\Lambda_{\mathbf{m},i} \equiv (\mathbf{m}\cdot\lambda) - \lambda_i \neq 0$ form a basis of $B^r = \mathbf{L_J}(H^r)$. The eigenvectors associated to the resonant eigenvalues, i.e., those such that $\Lambda_{\mathbf{m},i} = 0$, form a basis for the complementary subspace, G^r, of B^r in H^r.

Since $\lambda_1 = 0$, the resonant equations of order r (with $m_1 + m_2 + m_3 = r$) has unique solution

$$m_2 + \lambda_3 m_3 = 0 \Rightarrow m_1 = r, m_2 = m_3 = 0, \quad (18)$$

$$m_2 + \lambda_3 m_3 = 1 \Rightarrow m_1 = r-1, m_2 = 1, m_3 = 0, \quad (19)$$

$$m_2 + \lambda_3 m_3 = \lambda_3 \Rightarrow m_1 = r-1, m_2 = 0, m_3 = 1. \quad (20)$$

Equation (20) has another solution given by $m_1 = r - \lambda_3, m_2 = \lambda_3, m_3 = 0$, provided λ_3 is integer satisfying $1 < \lambda_3 \leq r$. This solution is discarded by our hypothesis on λ_3.

Then, $\left\{ x_1^r \mathbf{e}_1, x_1^{r-1} x_2 \mathbf{e}_2, x_1^{r-1} x_3 \mathbf{e}_3 \right\}$, form a basis for G^r in H^r.

By applying theorem A.1, we have that there exists a coordinate transformation $\mathbf{x} \rightarrow \mathbf{y}$, such that (10-12) has normal form (15-17) where a_r, b_r and c_r are some real constants. ■

The values of of all these constants can be uniquely determined, up to the desired order, by inductive construction in r. The idea is to introduce polynomial coordinate transformations $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{h}_r(\mathbf{x})$ such that $\mathbf{L_J}\mathbf{h}_r(\mathbf{x}) = \mathbf{X}_r(\mathbf{x})$+higher order terms. The higher order terms of the expansion are modified by successive coordinates transformations. Suppose we apply the r-th transformation, for a fixed r. Then, all non-resonant terms of order r, are eliminated. The r-th transformation modifies the expansion terms of order higher than r, but resonant terms of order less than r are not affected. One then removes the non-resonant terms of order $r+1$ by introducing the $(r+1)$-th transformation, and so on. In each step one obtains the desired coefficients. Finally, we can truncated the expansion up to order N.

We see that (15-17) satisfy the conditions of the existence and uniqueness theorem of differential equations. Then, there exist a unique solution of (15-17) passing through (y_{10}, y_{20}, y_{30}), at $t=0$. By neglecting the error

terms one is able to integrate the resulting approximated system

$$y_1' = \sum_{r=2}^{N} a_r y_1^r, \tag{21}$$

$$y_2' = y_2 \left(1 + \sum_{r=2}^{N} b_r y_1^{r-1}\right), \tag{22}$$

$$y_3' = y_3 \left(\lambda_3 + \sum_{r=2}^{N} c_r y_1^{r-1}\right), \tag{23}$$

with initial condition $(y_1(t_0), y_2(t_0), y_3(t_0)) = (y_{10}, y_{20}, y_{30})$.

The general solution of (21-23) is as follows.

If $y_{10} = 0$, then $y_1(t) = 0$, $y_2(t) = y_{20}e^{\tau-\tau_0}$ and $y_3(t) = y_{30}e^{\lambda_3(\tau-\tau_0)}$ for all $t \in \mathbb{R}$. Then, the orbit approach the origin as $\tau \to -\infty$ provided $\lambda_3 > 0$.

If $y_{10} \neq 0$, then (21-23) can be integrated in quadratures as

$$\tau - \tau_0 = \int_{y_{10}}^{y_1} \left(\sum_{r=2}^{N} a_r \zeta^r\right)^{-1} \mathrm{d}\zeta, \tag{24}$$

$$y_2(t) = y_{20} e^{\tau-\tau_0} \prod_{r=2}^{N} \exp\left[b_r \int_{\tau_0}^{\tau} y_1(t)^{r-1} \mathrm{d}t\right], \tag{25}$$

$$y_3(t) = y_{30} e^{\lambda_3(\tau-\tau_0)} \prod_{r=2}^{N} \exp\left[c_r \int_{\tau_0}^{\tau} y_1(t)^{r-1} \mathrm{d}t\right]. \tag{26}$$

The y_1-component of the orbit passing through (y_{10}, y_{20}, y_{30}) at $\tau = \tau_0$ with $y_{10} \neq 0$ is obtained by inverting the quadrature (24).

The other components are given by

$$y_2 = y_{20} \exp\left[\int_{y_{10}}^{y_1} \frac{1 + \sum_{r=2}^{N} b_r \zeta^{r-1}}{\sum_{r=2}^{N} a_r \zeta^r} \mathrm{d}\zeta\right], \tag{27}$$

$$y_3 = y_{30} \exp\left[\int_{y_{10}}^{y_1} \frac{1 + \sum_{r=2}^{N} c_r \zeta^{r-1}}{\sum_{r=2}^{N} a_r \zeta^r} \mathrm{d}\zeta\right]. \tag{28}$$

This orbit does not admit, in general, a prolongation to all the time values. If the maximal interval of definition, (α, β) of the solution y_1 is such that α is finite, then the orbits diverges in the y_1-direction as $\tau \to \alpha^+$.

A. Treatable case: normal expansion to third order for C_-

In this section we show normal form expansions for the vector field (10-12) defined in a vicinity of C_- expressed in the form of proposition IV.2. The proof is given in three steps. First, by applying linear coordinate transformations (translation to the origin and similarity transformation to reduce the Jacobian matrix to its real Jordan form) we reduce the linear part in its simpler form. Second, we perform a quadratic coordinate transformation in order to reduce non-resonant second order terms. Third, we perform a cubic coordinate transformation that allows to eliminate non-resonant terms of third order. In principle it is possible to eliminate all the non-resonant terms of all orders. However, we find the system computationally treatable to third order.

Proposition IV.2 *Let be the vector field* $\mathbf{X}$ *given by (10-12) which is* C^∞ *in a neighborhood of* $\mathbf{x}^* = (x_\phi^*, x_\varphi^*, y^*)^T \in C_-$. *Let* $m \geq n > 0$, *and* $x_\varphi^* \in \mathbb{R}$, *such that* $\lambda_3^- = 1 - n x_\phi^* + m\sqrt{1 + x_\phi^{*\,2}}$ *is not integer, then, there exist a transformation to new coordinates* $x \to z$, *such that (10-12), defined in a vicinity of* $\mathbf{x}^*$, *has normal form*

$$z_1' = O(|z|^4), \tag{29}$$

$$z_2' = z_2 + O(|z|^4), \tag{30}$$

$$z_3' = \left(\lambda_3^- + c_2 z_1 + c_3 z_1^2\right) z_3 + O(|z|^4), \tag{31}$$

where $c_2 = -n + \frac{m x_\varphi^*}{\sqrt{1 + x_\varphi^{*\,2}}}$ *and* $c_3 = -\frac{n x_\varphi^*}{2\left(1 + x_\varphi^{*\,2}\right)} + \frac{m}{2\sqrt{1 + x_\varphi^{*\,2}}}$.

Proof.

By using linear coordinates transformations the system (10-12) is reduced to

$$\mathbf{x}' = \mathbf{J}\mathbf{x} + \mathbf{X}_2(\mathbf{x}) + \mathbf{X}_3(\mathbf{x}) \tag{32}$$

where $\mathbf{x}$ stands for the phase vector $\mathbf{x} = (x_1,\, x_2,\, x_3)^T$, $\mathbf{J}$ stands for the Jordan Form of the matrix of derivatives

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - n x_\varphi^* + m\sqrt{1 + x_\varphi^{*\,2}} \end{pmatrix} \tag{33}$$

$$\mathbf{X}_2(\mathbf{x}) = \begin{pmatrix} X_{(1,1,0),1} x_1\, x_2 + X_{(0,0,2),1} x_3^2 \\ X_{(2,0,0),2} x_1^2 + X_{(0,2,0),2} x_2^2 + X_{(0,0,2),2} x_3^2 \\ X_{(1,0,1),3} x_1 x_3 + X_{(0,1,1),3} x_2 x_3 \end{pmatrix}, \tag{34}$$

where the coefficients $\mathbf{X}_{\mathbf{m},i}$ are displayed in table II for the allowed $\mathbf{m}$.

The third order terms become:

$$\mathbf{X}_3(\mathbf{x}) = \begin{pmatrix} X_{(3,0,0),1} x_1^3 + X_{(1,2,0),1} x_1 x_2^2 + X_{(1,0,2),1} x_1 x_3^2 \\ X_{(2,1,0),2} x_1^2 x_2 + X_{(0,3,0),2} x_2^3 + X_{(0,1,2),2} x_2 x_3^2 \\ X_{(2,0,1),3} x_1^2 x_3 + X_{(0,2,1),3} x_2^2 x_3 + X_{(0,0,3),3} x_3^3 \end{pmatrix}, \tag{35}$$

with the coefficients defined as in table III.

Second step: simplifying the quadratic part

TABLE II: Coefficients of the vector field $\mathbf{X}_2(\mathbf{x})$.

$\mathbf{m}$	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$X_{\mathbf{m},3}$
$(2,0,0)$	0	$-\frac{x_\varphi^*}{2\left(x_\varphi^{*\,2}+1\right)}$	0
$(1,1,0)$	$\frac{1}{x_\varphi^*}$	0	0
$(1,0,1)$	0	0	$-n+\frac{m x_\varphi^*}{\sqrt{1+x_\varphi^{*\,2}}}$
$(0,2,0)$	0	$\frac{3}{2x_\varphi^*}$	0
$(0,1,1)$	0	0	$-\frac{\left(-m\sqrt{1+x_\varphi^{*\,2}}+n x_\varphi^*-1\right)}{x_\varphi^*}$
$(0,0,2)$	$m x_\varphi^*\sqrt{1+x_\varphi^{*\,2}}-n\left(1+x_\varphi^{*\,2}\right)$	$\frac{1}{2}x_\varphi^*\left(-2m\sqrt{1+x_\varphi^{*\,2}}+2n x_\varphi^*-1\right)$	0

TABLE III: Coefficients of the vector field $\mathbf{X}_3(\mathbf{x})$.

$\mathbf{m}$	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$X_{\mathbf{m},3}$
$(3,0,0)$	$-\frac{1}{2\left(x_\varphi^{*\,2}+1\right)}$	0	0
$(2,1,0)$	0	$-\frac{1}{2\left(x_\varphi^{*\,2}+1\right)}$	0
$(2,0,1)$	0	0	$-\frac{1}{2\left(x_\varphi^{*\,2}+1\right)}$
$(1,2,0)$	$\frac{1}{2x_\varphi^{*\,2}}$	0	0
$(1,0,2)$	$-\frac{1}{2}$	0	0
$(0,3,0)$	0	$\frac{1}{2x_\varphi^{*\,2}}$	0
$(0,2,1)$	0	0	$\frac{1}{2x_\varphi^{*\,2}}$
$(0,1,2)$	0	$-\frac{1}{2}$	0
$(0,0,3)$	0	0	$-\frac{1}{2}$

TABLE IV: Eigenvalues of $\mathbf{L}^(2)_{\mathbf{J}} : H^2 \to H^2$.

$\mathbf{m}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},3}$
$(2,0,0)$	-	-1	-
$(1,1,0)$	1	-	-
$(1,0,1)$	-	-	0
$(0,2,0)$	-	1	-
$(0,1,1)$	-	-	1
$(0,0,2)$	$2\left(1-n x_\varphi^*+m\sqrt{1+x_\varphi^{*\,2}}\right)$	$1-2n x_\varphi^*+2m\sqrt{1+x_\varphi^{*\,2}}$	-

By the hypotheses the eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3^- = 1-n x_\varphi^{*\,2}+m\sqrt{1+x_\varphi^{*\,2}}$ of $\mathbf{J}$ are different. Hence, its eigenvectors form a basis of $\mathbb{R}^3$. The linear operator

$$\mathbf{L}_{\mathbf{J}}^{(2)} : H^2 \to H^2$$

has eigenvectors $\mathbf{x}^{\mathbf{m}}\mathbf{e}_i$ with eigenvalues $\Lambda_{\mathbf{m},i} = m_1\lambda_1 + m_2\lambda_2 + \lambda_3 m_3 - \lambda_i$, $i = 1,2,3$, $m_1, m_2, m_3 \geq 0$, m_1+m_2+

$m_3 = 2$. The eigenvalues $\Lambda_{\mathbf{m},i}$ for the available $\mathbf{m}, i$ are displayed in table IV. To obtain the normal form of (29-31) wee must look for resonant terms, i.e., those terms of the form $\mathbf{x}^{\mathbf{m}}\mathbf{e}_i$ with $\mathbf{m}$ and i such that $\Lambda_{\mathbf{m},i} = 0$ for the available $\mathbf{m}, i$. Only one term is resonant of second order: $\Lambda_{(1,0,1),3} = 0 \rightarrow c_2 y_1 y_3 \mathbf{e}_3$.

The required function

$$\mathbf{h}_2 : H^2 \rightarrow H^2$$

to eliminate the non-resonant quadratic terms is given by

$$\mathbf{h}_2(\mathbf{y}) = \begin{pmatrix} \frac{X_{(1,1,0),1}}{\Lambda_{(1,1,0),1}} y_1\, y_2 + \frac{X_{(0,0,2),1}}{\Lambda_{(0,0,2),1}} y_3^2 \\ \frac{X_{(2,0,0),2}}{\Lambda_{(2,0,0),2}} y_1^2 + \frac{X_{(0,2,0),2}}{\Lambda_{(0,2,0),2}} y_2^2 + \frac{X_{(0,0,2),2}}{\Lambda_{(0,0,2),2}} y_3^2 \\ \frac{X_{(0,1,1),3}}{\Lambda_{(0,1,1),3}} y_2 y_3 \end{pmatrix}, \tag{36}$$

The quadratic transformation

$$\mathbf{x} \rightarrow \mathbf{y} + \mathbf{h}_2(\mathbf{y}) \tag{37}$$

with $\mathbf{h}_2$ defined as in (36) is the coordinate transformation required in theorem A.1. By applying this theorem we prove the existence of the required constant c_2.

To finish the proof, let us calculate the value of c_2.

By applying the transformation (37) the vector field (32) transforms to

$$\mathbf{y}' = \mathbf{J}\mathbf{y} - \mathbf{L}_{\mathbf{J}}^{(2)}\mathbf{h}_2(\mathbf{y}) + \mathbf{X}_2(\mathbf{y}) + \tilde{\mathbf{X}}_{\mathbf{3}}(\mathbf{y}) + \mathbf{O}(|\mathbf{y}|^{\mathbf{4}}), \tag{38}$$

Since

$$-\mathbf{L}_{\mathbf{J}}^{(2)}\mathbf{h}_2(\mathbf{y}) + \mathbf{X}_2(\mathbf{y}) = X_{(1,0,1),3} y_1 y_3 \mathbf{e}_3, \tag{39}$$

we have

$$\mathbf{y}' = \mathbf{J}\mathbf{y} + X_{(1,0,1),3} y_1 y_3 \mathbf{e}_3 + \tilde{\mathbf{X}}_3(\mathbf{y}) + \mathbf{O}(|\mathbf{y}|^4) \tag{40}$$

, i.e., $c_2 = X_{(1,0,1),3} = -n + \frac{m x_\varphi^*}{\sqrt{1+{x_\varphi^*}^2}}$.

The vector field $\tilde{\mathbf{X}}_3(\mathbf{y})$ introduced above has the coefficients:

$$\begin{aligned}
\tilde{X}_{(2,0,1),3} &= \frac{m}{2\sqrt{{x_\varphi^*}^2 + 1}} - \frac{n x_\varphi^*}{2\left({x_\varphi^*}^2 + 1\right)}, \\
\tilde{X}_{(1,2,0),1} &= \frac{3}{{x_\varphi^*}^2}, \\
\tilde{X}_{(1,0,2),1} &= -\frac{n^2\delta^2 + m\left(\delta + m\left(\delta^2 - 1\right)\right) - n x_\varphi^*[2\mathrm{m}\delta + 1] + 1}{(\lambda_3)^-}, \\
\tilde{X}_{(1,0,2),2} &= \frac{x_\varphi^*\left[2x_\varphi^* m^2 + \frac{\left(x_\varphi^* - 2\left(2\left(\delta^2-1\right)n + n\right)\right)m}{\delta} + n\left(2n x_\varphi^* - 1\right)\right]}{2\lambda_3^-}, \\
\tilde{X}_{(0,3,0),2} &= \frac{5}{{x_\varphi^*}^2}, \\
\tilde{X}_{(0,2,1),3} &= \frac{\left(-\mathrm{m}\delta + n x_\varphi^* - 2\right)\left(-2\mathrm{m}\delta + 2n x_\varphi^* - 3\right)}{2{x_\varphi^*}^2}, \\
\tilde{X}_{(0,1,2),1} &= \frac{\delta\left(4\delta^2 x_\varphi^* m^3 + 4\delta\Delta_1 m^2 + \Delta_2 m + \mathrm{n}\delta\Delta_3\right)}{2\lambda_3^- x_\varphi^*}, \\
\tilde{X}_{(0,1,2),2} &= -2\left(\delta^2 - 1\right)n^2 + x_\varphi^*[4\mathrm{m}\delta + 3]n - \mathrm{m}\delta(2\mathrm{m}\delta + 3) - 3, \\
\tilde{X}_{(0,0,3),3} &= -4n x_\varphi^*\left[n x_\varphi^* - 2\right] - 5,
\end{aligned} \tag{41}$$

where $\delta = \sqrt{{x_\varphi^*}^2 + 1}$, $\Delta_1 = 2x_\varphi^* - n\left(3{x_\varphi^*}^2 + 1\right)$, $\Delta_2 = 4n\left(-4{x_\varphi^*}^2 + n\left(3{x_\varphi^*}^2 + 2\right)x_\varphi^* - 2\right) + 5x_\varphi^*$, $\Delta_3 = -4n x_\varphi^*\left[n x_\varphi^* - 2\right] - 5$.

Third step: simplifying the cubic part

After the last two steps, the equation (32) is transformed to

$$\mathbf{y}' = \mathbf{J}\mathbf{y} + c_2 y_1 y_3 \mathbf{e}_3 + \tilde{\mathbf{X}}_3(\mathbf{y}) + \mathbf{O}(|\mathbf{y}|^4). \tag{42}$$

There is only one term of order three which is resonant (see table V): $\Lambda_{(2,0,1),3} = 0 \rightarrow c_3 z_1^2 z_3 \mathbf{e}_3$.

As in the last step, in order to eliminate non-resonant terms of third order we will consider the coordinate trans-

TABLE V: Eigenvalues of $\mathbf{L}_{\mathbf{J}}^{(3)} : H^3 \to H^3$.

$\mathbf{m}$	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},2}$	$\Lambda_{\mathbf{m},3}$
$(2,0,1)$	-	-	0
$(1,2,0)$	2	-	-
$(1,0,2)$	$2\left(m\sqrt{1+x_\varphi^{*\,2}}-nx_\varphi^*+1\right)$	$2m\sqrt{1+x_\varphi^{*\,2}}-2nx_\varphi^*+1$	-
$(0,3,0)$	-	2	-
$(0,2,1)$	-	-	2
$(0,1,2)$	$2m\sqrt{1+x_\varphi^{*\,2}}-2nx_\varphi^*+3$	$2\left(m\sqrt{1+x_\varphi^{*\,2}}-nx_\varphi^*+1\right)$	-
$(0,0,3)$	-	-	$2\left(m\sqrt{1+x_\varphi^{*\,2}}-nx_\varphi^*+1\right)$

formation $\mathbf{y} \to \mathbf{z}$ given by

$$\mathbf{y} = \mathbf{z} + \mathbf{h}_3(\mathbf{z}) \tag{43}$$

where

$$\mathbf{h}_3 : H^3 \to H^3$$

is defined by

$$\mathbf{h}_3(\mathbf{z}) = \begin{pmatrix} \frac{\tilde{X}_{(1,2,0),1}}{\Lambda_{(1,2,0),1}} z_1 z_2^2 + \frac{\tilde{X}_{(1,0,2),1}}{\Lambda_{(1,0,2),1}} z_1 z_3^2 + \frac{\tilde{X}_{(0,1,2),1}}{\Lambda_{(0,1,2),1}} z_2 z_3^2 \\ \frac{\tilde{X}_{(1,0,2),2}}{\Lambda_{(1,0,2),2}} z_1 z_3^2 + \frac{\tilde{X}_{(0,3,0),2}}{\Lambda_{(0,3,0),2}} z_2^3 + \frac{\tilde{X}_{(0,1,2),2}}{\Lambda_{(0,1,2),2}} z_2 z_3^2 \\ \frac{\tilde{X}_{(0,2,1),3}}{\Lambda_{(0,2,1),3}} z_2^2 z_3 + \frac{\tilde{X}_{(0,0,3),3}}{\Lambda_{(0,0,3),3}} z_3^3 \end{pmatrix}, \tag{44}$$

where $\Lambda_{\mathbf{m},i}$ are the eigenvalues of the operator linear operator

$$\mathbf{L}_{\mathbf{J}}^{(3)} : H^3 \to H^3$$

associated to the eigenvectors $\mathbf{x}^{\mathbf{m}}\mathbf{e}_i$. In table V are shown these eigenvalues. The associated eigenvectors form a basis of H^3 (the space of vector fields with polynomial components of third degree) because the eigenvalues of $\mathbf{J}$ are different.

The transformation (43) is the required by theorem (A.1). By using this theorem we prove the existence of the required constant c_3. To find its we must to calculate

$$-\mathbf{L}_{\mathbf{J}}^{(3)}\mathbf{h}_3(\mathbf{z}) + \tilde{\mathbf{X}}_3(\mathbf{z}),$$

which it is equal to

$$\tilde{X}_{(2,0,1),3} z_1^3 \mathbf{e}_3$$

where $\mathbf{e}_i,\ i = 1,2,3$, is the canonical basis in $\mathbb{R}^n$. Then,

$$c_3 = \frac{m}{2\sqrt{1+x_\varphi^{*\,2}}} - \frac{nx_\varphi^*}{2\left(x_\varphi^{*\,2}+1\right)}.$$

Observe that the transformation $\mathbf{h}_3$ does not affect the value of the coefficient of the resonant term of order $r = 2$. Then, the result of the proposition follows. ■

1. *Unstable manifold to third order*

For $\lambda_3^- > 0$, the origin has a 2-dimensional local unstable manifold tangent to the plane z_2-z_3 at $\mathbf{0}$ given by

$$W^u_{loc}(\mathbf{0}) = \{(z_1,z_2,z_3) \in \mathbb{R}^3 : z_1 = h(z_2,z_2), \quad \mathbf{Dh}(\mathbf{0}) = \mathbf{0}, \left|(z_2,z_3)^T\right| < \delta\}, \tag{45}$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is a C^r function and $\delta > 0$ is small enough.

Using the invariance of $W^u_{loc}(\mathbf{0})$ under the dynamics of (29-31) we obtain a quasi-linear partial differential equation that h must satisfy:

$$\mathcal{N}\left(h\left(z_2,z_3\right)\right) = \mathbf{O}(|\mathbf{z}|^4) \tag{46}$$

where we have defined the differential operator

$$\mathcal{N}\left(h\left(z_2,z_3\right)\right) \equiv z_2\frac{\partial h}{\partial z_2} + z_3\left(c_3h^2 + c_2h + \lambda_3^-\right)\frac{\partial h}{\partial z_3}. \tag{47}$$

This system is solved up to order $\mathbf{O}(|\mathbf{z}|^4)$.

Assuming that $W^c_{\mathrm{loc}}(\mathbf{0})$ is $C^4\left(\mathbb{R}^3\right)$ we can expressed it as the graph

$$h\left(z_2,z_3\right) \equiv h_{3,0}z_2^3 + h_{2,0}z_2^2 + z_3h_{2,1}z_2^2 + z_3h_{1,1}z_2 + z_3^2h_{1,2}z_2 + z_3^2h_{0,2} + z_3^3h_{0,3} + \mathbf{O}(|\mathbf{z}|^4). \tag{48}$$

This function satisfy the tangentiality conditions $h(0,0) = \partial_{z_2}h(0,0) = \partial_{z_3}h(0,0) = (0,0)$.

By substitution in the differential equation $\mathcal{N}\left(h\left(z_2,z_3\right)\right) = 0$ and by discarding the error terms we find

$$3h_{3,0}z_2^3+2h_{2,0}z_2^2+h_{2,1}z_3\left(\lambda_3^-+2\right)z_2^2+h_{1,1}z_3\left(\lambda_3^-+1\right)z_2+h_{1,2}z_3^2\left[2\lambda_3^-+1\right]z_2+2z_3^2h_{0,2}\lambda_3^-+3z_3^3h_{0,3}\lambda_3^-=0 \quad (49)$$

Using the condition $\lambda_3^- > 0$ we get $h_{3,0} = h_{2,1} = h_{2,0} = h_{1,2} = h_{1,1} = h_{0,3} = h_{0,2} = 0$. Then, the unstable manifold of the origin is, up to order $O(|\mathbf{z}|^4)$, $W^u_{loc}(\mathbf{0}) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 = 0, z_2^2 + z_3^2 < \delta^2\}$ where δ is a real value small enough. Therefore, the dynamics of (29-31) restricted to the unstable manifold, is given, up to order $\mathbf{O}(|\mathbf{z}|^4)$, by $z_1 \equiv 0$, $z_2(\tau) = e^{\tau} z_{20}$, $z_2(\tau) = e^{\lambda_3^- \tau} z_{30}$, where $z_{20}^2 + z_{30}^2 < \delta^2$. This means that $\lim_{\tau\to-\infty}(z_1(\tau),\, z_2(\tau),\, z_3(\tau)) = (0,0,0)$. Then, the origin is the past attractor for an open set of orbits of (29-31). In the original coordinates this means that the past asymptotic dynamics of an open set of orbits of the system can be accurately approximated by a massless quintom model located at C_-. Let us investigate the center manifold.

2. *Center manifold to third order*

For $\lambda_3^- > 0$, the origin has a 1-dimensional local unstable manifold tangent to the plane z_1 at $\mathbf{0}$ given by

$$\begin{aligned} W^c_{loc}(\mathbf{0}) = \{&(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2 = f(z_1), z_3 = g(z_1),\\ &Df(0) = 0, Dg(0) = 0, |z_1| < \delta\}, \end{aligned} \tag{50}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are C^r functions of z_1 and $\delta > 0$ is small enough.

Using the invariance de $W^c_{loc}(\mathbf{0})$ under the dynamics of (29-31) we obtain a system of two ordinary differential equations that f and g must satisfy:

$$f(z_1) - f'(z_1)O\left(|\mathbf{z}|^4\right) = O\left(|\mathbf{z}|^4\right), \tag{51}$$

$$\left(\lambda_3^- + c_2 z_1 + c_3 z_1^2\right) g(z_1) - g'(z_1)O\left(|\mathbf{z}|^4\right) = O\left(|\mathbf{z}|^4\right) \tag{52}$$

In order to obtain an expression for the center manifold of the origin up to order $\mathbf{O}\left(|\mathbf{z}|^4\right)$ we need to solve equations (51-52) up to the same order. Assuming that $W^c_{loc}(\mathbf{0})$ is C^4, it can be expressed as

$$f(z_1) = a z_1^2 + b z_1^3 + O\left(|z_1|^4\right), \tag{53}$$

$$g(z_1) = c z_1^2 + d z_1^3 + O\left(|z_1|^4\right). \tag{54}$$

The representation (53-54) satisfy the tangentiality conditions $f(0) = g(0) = f'(0) = g'(0)$. By substitution of (53-54) in (51-52) and neglecting error terms we get

$$a z_1^2 + b z_1^3 = O\left(|z_1|^4\right) \implies a = b = 0, \tag{55}$$

$$\begin{aligned} &\left(c c_2 + d\lambda_3^-\right) z_1^3 + c\lambda_3^- z_1^2 = O\left(|z_1|^4\right)\\ &\implies c\lambda_3^- = c c_2 + d\lambda_3^- = 0\\ &\implies c = d = 0. \end{aligned} \tag{56}$$

The last implication follows from the fact that $\lambda_3^- > 0$. Then, the center manifold of the origin is, up to order $O(|\mathbf{z}|^4)$, $W^c_{loc}(\mathbf{0}) = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2 = z_3 = 0, |z_1| < \delta\}$ where δ is a real value small enough.

Let us describe the center manifold, up to the prescribed order, as a graph in the original variables.

We have

$$x_1 \equiv x_\phi + \sqrt{{x_\varphi^*}^2 + 1} = -\frac{z_1\left(z_1 + 2x_\varphi^*\right)}{2\sqrt{{x_\varphi^*}^2 + 1}} + O\left(|z_1|^4\right), \tag{57}$$

$$x_2 \equiv x_\varphi - x_\varphi^* = \frac{x_\varphi^* z_1^2}{2{x_\varphi^*}^2 + 2} + z_1 + O\left(|z_1|^4\right), \tag{58}$$

$$x_3 \equiv y = O\left(|z_1|^4\right). \tag{59}$$

By taking the inverse, up to fourth order, of (58) we have the expression for z_1:

$$z_1 = \frac{{x_\varphi^*}^2 x_2^3}{2\left({x_\varphi^*}^2 + 1\right)^2} - \frac{x_\varphi^* x_2^2}{2\left({x_\varphi^*}^2 + 1\right)} + x_2 + O\left(|x_2|^4\right). \tag{60}$$

Substituting (60) in (57-59) we have that the center manifold of the origin is given by the graph: $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = \frac{x_\varphi^* x_2^3}{2({x_\varphi^*}^2+1)^{5/2}} - \frac{x_2^2}{2({x_\varphi^*}^2+1)^{3/2}} - \frac{x_\varphi^* x_2}{\sqrt{{x_\varphi^*}^2+1}} + O\left(|x_2|^4\right), x_3 = O\left(|x_2|^4\right), |x_2| < \delta\}$ for $\delta > 0$ small enough.

In conclusion, the analysis up to fourth order does not allows to classify the global past asymptotic dynamics of quintom model. Since the numerical experiments in [7] suggest that there is an open set of orbits that tends to infinity, then it is worthy to investigate the dynamics at infinity.

V. ANALYSIS AT INFINITY

In this section we investigate the dynamics at infinity by using the central Poincaré projection method [31].

To obtain the critical points at infinity we introduce spherical coordinates (ρ is the inverse of $r = \sqrt{x_\phi^2 + x_\varphi^2 + y^2}$, then, $\rho \to 0$ as $r \to \infty$):

$$x_\phi = \frac{1}{\rho}\sin\theta_1\cos\theta_2, \tag{61}$$

$$y = \frac{1}{\rho}\sin\theta_1\sin\theta_2, \tag{62}$$

$$x_\varphi = \frac{1}{\rho}\cos\theta_1 \tag{63}$$

TABLE VI: Location and existence conditions for the critical points at infinity.

Name	θ_1	θ_2	Existence
$P_1^{\pm}$	0	$\pm\frac{\pi}{2}$	always
$P_2^{\pm}$	π	$\pm\frac{\pi}{2}$	always
$P_3^{\pm}$	$\frac{\pi}{4}$	$\pm\cos^{-1}\left(-\frac{m}{n}\right)$	$-\pi < \pm\cos^{-1}\left(-\frac{m}{n}\right) \le \pi, n \neq 0$
$P_4^{\pm}$	$\frac{3\pi}{4}$	$\pm\cos^{-1}\left(\frac{m}{n}\right)$	$-\pi < \pm\cos^{-1}\left(\frac{m}{n}\right) \le \pi, n \neq 0$
P_5	$\theta_1^\star$	0	$0 \le \theta_1^\star \le \pi$
P_6	$\theta_1^\star$	π	$0 \le \theta_1^\star \le \pi$

TABLE VII: Stability of the critical points at infinity. We use the notation $\delta = m^2 - n^2$ and $\lambda^{\pm} = n\cos\theta_1^\star \pm m\sin\theta_1^\star$.

Name	(λ_1, λ_2)	ρ'	Stability
$P_1^{\pm}$	$(-n, n)$	> 0	saddle
$P_2^{\pm}$	$(-n, n)$	> 0	saddle
$P_3^{\pm}$	$\left(\frac{\sqrt{2}\delta}{n}, \frac{\delta}{\sqrt{2}n}\right)$	$\begin{cases} > 0, & \delta < 0 \\ < 0, & \delta > 0 \end{cases}$	source if $n < 0, n < m < -n$ saddle otherwise
$P_4^{\pm}$	$\left(-\frac{\sqrt{2}\delta}{n}, -\frac{\delta}{\sqrt{2}n}\right)$	$\begin{cases} > 0, & \delta < 0 \\ < 0, & \delta > 0 \end{cases}$	source if $n > 0, -n < m < n$ saddle otherwise
P_5	$\left(0, \lambda^{+}\right)$	$\begin{cases} < 0, & \frac{\pi}{4} < \theta_1^\star < \frac{3\pi}{4} \\ > 0, & \text{otherwise} \end{cases}$	nonhyperbolic
P_6	$\left(0, \lambda^{-}\right)$	$\begin{cases} < 0, & \frac{\pi}{4} < \theta_1^\star < \frac{3\pi}{4} \\ > 0, & \text{otherwise} \end{cases}$	nonhyperbolic

where $0 \le \theta_1 \le \pi$ and $-\pi < \theta_2 \le \pi$, and $0 < \rho < \infty$.

Defining the time derivative $f' \equiv \rho \mathrm{d}f/\mathrm{d}\tau$, the system (10-12), can be written as

$$\rho' = \frac{1}{2}\left(\cos^2\theta_1 - \cos(2\theta_2)\sin^2\theta_1\right) + 2n\cos\theta_1\sin^2\theta_1\sin^2\theta_2\rho + O\left(\rho^2\right). \tag{64}$$

and

$$\begin{aligned} \theta_1' &= n\cos(2\theta_1)\sin\theta_1\sin^2\theta_2 - \cos\theta_1\sin\theta_1\sin^2\theta_2\rho + O\left(\rho^2\right), \\ \theta_2' &= (n\cos\theta_1\cos\theta_2 + m\sin\theta_1)\sin\theta_2 - \cos\theta_2\sin\theta_2\rho + O\left(\rho^2\right). \end{aligned} \tag{65}$$

Since equation (64) does not depends of the radial component at the limit $\rho \to 0$, we can obtain the critical points at infinity by solving equations (65) in the limit $\rho \to 0$. Thus, the critical points at infinite must satisfy the compatibility conditions

$$\cos(2\theta_1)\sin\theta_1\sin^2\theta_2 = 0,$$
$$(n\cos\theta_1\cos\theta_2 + m\sin\theta_1)\sin\theta_2 = 0. \quad (66)$$

First, we examine the stability of the pairs $(\theta_1^\star, \theta_2^\star)$ satisfying the compatibility conditions (66) in the plane θ_1-θ_2, and then, we examine the global stability by substituting in (64) and analyzing the sign of $\rho'(\theta_2^\star, \theta_2^\star)$. In table VI it is offered information about the location and existence conditions of these critical points. In table VII we summarize the stability properties of these critical points.

The cosmological solutions associated to the critical points $P_1^\pm$ and $P_2^\pm$ have the evolution rates $\dot\phi^2/V = 0, \dot\phi/\dot\varphi = 0$ and $H/\dot\varphi \equiv \rho/\sqrt{6} \to 0$. [1] These solutions are always saddle points at infinity. The critical points $P_3^\pm$ and $P_4^\pm$ are sources provided $n < 0, n < m < -n$ or $n > 0, -n < m < n$, respectively. They are saddle points otherwise. The associated cosmological solutions to $P_3^\pm$ have the evolution rates $\dot\phi^2/V = \frac{2m^2}{n^2-m^2}$, $\dot\phi/\dot\varphi = -m/n$, and $H/\dot\phi \equiv -n\rho/(\sqrt{3}m) \to 0$, and $H/\dot\varphi \equiv \rho/\sqrt{3} \to 0$, whereas the associated cosmological solutions to $P_4^\pm$ have the evolution rates $\dot\phi^2/V = \frac{2m^2}{n^2-m^2}$, $\dot\phi/\dot\varphi = -m/n$, and $H/\dot\phi \equiv n\rho/(\sqrt{3}m) \to 0$, and $H/\dot\varphi \equiv -\rho/\sqrt{3} \to 0$. The curves of critical points P_5 and P_6 are nonhyperbolic. The associated cosmological solutions have expansion rates (valid for $\theta_1^\star \neq \pi/4$) $V/\dot\phi^2 = 0, \dot\phi/\dot\varphi = \tan\theta_1^\star, H/\dot\varphi = \rho\sec\theta_1^\star/\sqrt{6} \to 0$, and $V/\dot\phi^2 = 0, \dot\phi/\dot\varphi = -\tan\theta_1^\star, H/\dot\varphi = \rho\sec\theta_1^\star/\sqrt{6} \to 0$, respectively.

Concluding, although there exists unbounded orbits towards the past, by examining the orbits at infinity, we get that the sources satisfy the evolution rates $\dot\phi^2/V \sim \frac{2m^2}{n^2-m^2}$, $\dot\phi/\dot\varphi \sim -m/n$, with $H/\dot\phi \to 0$, provided $n < 0, n < m < -n$ or $n > 0, -n < m < n$.

VI. CONCLUDING REMARKS

In this paper we have investigated the past asymptotic dynamics of quintom cosmologies (which is closely related with the behavior of typical orbits near an hyperbolae of critical points) by using the Normal Form Theorem as a tool. We have developed the general (arbitrary order) normal expansion for the quintom model. This expansion is the minimal in the sense that the terms that are involved are the "essential" degrees of nonlinearity. The other higher order terms are removed by making properly coordinate transformation. By integrating the resulting system one is able to construct unstable and center manifolds. We have explored the structure of unstable and center manifold of the origin up to fourth order. From the structure of the unstable manifold we see that that the past asymptotic behavior of quintom cosmologies, with exponential potentials, is given by a massless scalar field cosmology for an open set of orbits. We provide here an approximated formula for the center manifold. By examining the center manifold we find that, up to fourth order, the curve of critical points C_- is not the past asymptotic global attractor.

To get more accuracy we need to deal with the expansion up to arbitrary order $N > 3$. However, it is very difficult to get the expansion coefficients for $N > 3$. Theoretically, if the solution of (21) for y_1 admits a prolongation to $-\infty < \tau < \infty$, (for instance, if $y_{10} = 0$), then the trajectory passing through (y_{10}, y_{20}, y_{30}) at t_0 approaches the origin as $\tau \to -\infty$, provided $\lambda_3 > 0$. Otherwise, if the maximal interval, (α, β), has α finite, then, the orbits diverges in a finite time has $\tau \to \alpha^+$. This fact is supported by the numerical simulations in reference [7] (it appears that an open set of orbits escape to infinity into the past, which means that the local sources could not be the past attractor).

By examining the dynamics at infinity, using the central Poincaré projection method, we get that the sources satisfy the evolution rates $\dot\phi^2/V \sim \frac{2m^2}{n^2-m^2}$, $\dot\phi/\dot\varphi \sim -m/n$, with H an infinitesimal of $\dot\phi$, provided $n < 0, n < m < -n$ or $n > 0, -n < m < n$. These results complete the past asymptotic analysis of quintom cosmologies.

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APPENDIX A: NORMAL FORMS FOR VECTOR FIELDS

In this section we offer the main techniques for the construction of normal forms for vector fields in $\mathbb{R}^n$. We follow the approach in [27].

Let $\mathbf{X} : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field satisfying $\mathbf{X}(\mathbf{0}) = \mathbf{0}$. We can formally construct the Taylor expansion of $\mathbf{x}$ about $\mathbf{0}$, namely, $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_k + O(|\mathbf{x}|^{k+1})$, where $\mathbf{X}_r \in H^r$, the real vector space of vector fields whose components are homogeneous polynomials of degree r. For $r = 1$ to k we write

$$\mathbf{X}_r(\mathbf{x}) = \textstyle\sum_{m_1=1}^r \ldots \sum_{m_n=1}^r \sum_{j=1}^n \mathbf{X}_{\mathbf{m},j}\mathbf{x}^{\mathbf{m}}\mathbf{e}_j,$$
$$\textstyle\sum_i m_i = r, \quad (A1)$$

Observe that $\mathbf{X}_1 = \mathbf{DX}(\mathbf{0})\mathbf{x} \equiv \mathbf{A}\mathbf{x}$, i.e., the matrix of derivatives.

The aim of the normal form calculation is to construct a sequence of transformations which successively remove the non-linear term $\mathbf{X}_r$, starting from $r = 2$.

The transformation themselves are of the form

$$\mathbf{x} = \mathbf{y} + \mathbf{h}_r(\mathbf{y}), \quad (A2)$$

[1] Do not confuse ρ with the matter energy density, the latter denoted by ρ_{M}.

where $\mathbf{h}_r \in H^r$, $r \geq 2$.

The effect of (A2) in $\mathbf{X}_1$ is as follows [27]: Observe that $\mathbf{x} = O(|\mathbf{y}|)$. Then, the inverse of (A2) takes the form

$$\mathbf{y} = \mathbf{x} - \mathbf{h}_r(\mathbf{x}) + O(|\mathbf{x}|^{r+1}). \tag{A3}$$

By applying total derivatives in both sides, and assuming $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{X}_r(\mathbf{x})$, we find

$$\mathbf{y}' = \mathbf{A}\mathbf{y} - \mathbf{L}_{\mathbf{A}}\mathbf{h}_r(\mathbf{y}) + \mathbf{X}_r(\mathbf{y}) + O(|\mathbf{y}|^{r+1}) \tag{A4}$$

where $\mathbf{L}_{\mathbf{A}}$ is the linear operator that assigns to $\mathbf{h}(\mathbf{y}) \in H^r$ the Lie bracket of the vector fields $\mathbf{A}\mathbf{y}$ and $\mathbf{h}(\mathbf{y})$:

$$\begin{aligned} \mathbf{L}_{\mathbf{A}} : H^r &\to H^r \\ \mathbf{h} &\to \mathbf{L}_{\mathbf{A}}\mathbf{h}(\mathbf{y}) = \mathbf{D}\mathbf{h}(\mathbf{y})\mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{h}(\mathbf{y}). \end{aligned} \tag{A5}$$

Both $\mathbf{L}_{\mathbf{A}}$ and $\mathbf{X}_r \in H^r$, so that the deviation of the right-hand side of (A4) from $\mathbf{A}\mathbf{y}$ has no terms of order less than r in $|\mathbf{y}|$. This means that if $\mathbf{X}$ is such that $\mathbf{X}_2 = \ldots \mathbf{X}_{r-1} = 0$, they will remain zero under the transformation (A2). This makes clear how we may be able to remove $\mathbf{X}_r$ from a suitable choice of $\mathbf{X}_r$.

The proposition 2.3.2 in [27] states that if the inverse of $\mathbf{L}_{\mathbf{A}}$ exists, the differential equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{X}_r(\mathbf{x}) + O(|\mathbf{x}|^{r+1}) \tag{A6}$$

with $\mathbf{X}_r \in H^r$, it is transformed to

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + O(|\mathbf{y}|^{r+1}) \tag{A7}$$

by the transformation (A2) where

$$\mathbf{h}_r(\mathbf{y}) = \mathbf{L}_{\mathbf{A}}{}^{-1}\mathbf{X}_r(\mathbf{y}) \tag{A8}$$

The equation

$$\mathbf{L}_{\mathbf{A}}\mathbf{h}_r(\mathbf{y}) = \mathbf{X}_r(\mathbf{y}) \tag{A9}$$

is named the homological equation.

If $\mathbf{A}$ has distinct eigenvalues λ_i, $i = 1, 2, 3$, its eigenvectors form a basis of $\mathbb{R}^n$. Relative to this eigenbasis, $\mathbf{A}$ is diagonal. It can be proved (see proof in [27]) that $\mathbf{L}_{\mathbf{A}}$ has eigenvalues $\Lambda_{\mathbf{m},i} = \mathbf{m} \cdot \lambda - \lambda_i = \sum_j m_j\lambda_j - \lambda_i$ with associated eigenvectors $\mathbf{x}^{\mathbf{m}}\mathbf{e}_i$. The operator, $\mathbf{L}_{\mathbf{A}}{}^{-1}$, exists if and only if the $\Lambda_{\mathbf{m},i} \neq 0$, for every allowed $\mathbf{m}$ and $i = 1 \ldots r$.

If we were able to remove all the nonlinear terms in this way, then the vector field can be reduced to its linear part

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}) \to \mathbf{y}' = \mathbf{A}\mathbf{y}.$$

Unfortunately, not all the higher order terms vanishes by applying this transformations. Particularly if resonance occurs.

The n-tuple of eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ is resonant of order r (see definition 2.3.1 in [27]) if there exist some $\mathbf{m} = (m_1, m_2, \ldots m_n)^T$ (a n-tuple of non-negative integers) with $m_1 + m_2 + \ldots m_n = r$ and some $i = 1 \ldots n$ such that $\lambda_i = \mathbf{m} \cdot \lambda$, i.e., if $\Lambda_{\mathbf{m},i} = 0$ for some $\mathbf{m}$ and some i.

If there is no resonant eigenvalues, and provided they are different, we can use the eigenvectors of $\mathbf{A}$ as a basis for H^r. Then, we can write $\mathbf{h}_r$ as

$$\mathbf{h}_r(\mathbf{x}) = \sum_{\mathbf{m},i,\sum m_j = r} h_{\mathbf{m},i}\mathbf{x}^{\mathbf{m}}\mathbf{e}_i$$

and any vector field $\mathbf{X} \in H^r$ as

$$\mathbf{X}(\mathbf{x}) = \sum_{\mathbf{m},i,\ \sum m_j = r} \mathbf{X}_{\mathbf{m},i}\mathbf{x}^{m}\mathbf{e}_i$$

where $\mathbf{m} = (m_1, m_2, \ldots, m_n)^T$, $\mathbf{x}^m = x_1^{m_1}\, x_2^{m_2}\, \ldots\, x_n^{m_n}$ and $\mathbf{e}_i$, $i = 1, \ldots n$ stands for the canonical basis in $\mathbb{R}^n$. If the eigenvalues of $\mathbf{A}$ are not resonant of order r, then

$$h_{\mathbf{m},i} = X_{\mathbf{m},i}/\Lambda_{\mathbf{m},i}.$$

This gives $\mathbf{h}_r$ explicitly in terms of $\mathbf{X}_r$.

In the case of resonance occurs, we proceed as follows. If $\mathbf{A}$ can diagonalized, then the eigenvectors of $\mathbf{L}_{\mathbf{A}}$ form a basis of H^r. The subset of eigenvectors of $\mathbf{L}_{\mathbf{A}}$ with non-zero eigenvalues then form a basis of the image, B^r, of H^r under $\mathbf{L}_{\mathbf{A}}$. It follows that the component of $\mathbf{X}_r$ in B^r can be expanded in terms of these eigenvectors and $\mathbf{h}_r$ chosen such that

$$h_{\mathbf{m},i} = X_{\mathbf{m},i}/\Lambda_{\mathbf{m},i}.$$

to ensure the removal of these terms. The component, $\mathbf{w}_r$, of $\mathbf{X}_r$ lying in the complementary subspace, G^r, of B^r in H^r will be unchanged by the transformations $\mathbf{x} = \mathbf{y} + \mathbf{h}_r(\mathbf{y})$ obtained from B^r.

Since

$$\mathbf{X}_r(\mathbf{y} + \mathbf{h}_{r+k}(\mathbf{y})) = \mathbf{X}_r(\mathbf{y}) + O(|\mathbf{y}|^{r+k+1}), r \geq 2, k = 1, 2, \ldots,$$

these terms are not changed by subsequent transformations to remove non-resonant terms of higher order.

The above facts are expressed in

Theorem A.1 (theorem 2.3.1 in [27]) *Given a smooth vector field $\mathbf{X}(\mathbf{x})$ on $\mathbb{R}^n$ with $\mathbf{X}(\mathbf{0}) = \mathbf{0}$, there is a polynomial transformation to new coordinates, $\mathbf{y}$, such that the differential equation $\mathbf{x}' = \mathbf{X}(\mathbf{x})$ takes the form $\mathbf{y}' = \mathbf{J}\mathbf{y} + \sum_{r=1}^{N}\mathbf{w}_r(\mathbf{y}) + O(|\mathbf{y}|^{N+1})$, where $\mathbf{J}$ is the real Jordan form of $\mathbf{A} = \mathbf{D}\mathbf{X}(\mathbf{0})$ and $\mathbf{w}_r \in G^r$, a complementary subspace of H^r on $B^r = \mathbf{L}_{\mathbf{A}}(H^r)$.*

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